Example 1

1. *Derive the adding formula for combining the reflection R and transmission T matrices from the interaction principles for two atmospheric layers.*

We write down the interaction principles for the two layers

$$I_0^+ = T_1^+ I_1^+ + R_1^+ I_0^- + S_1^+ \qquad I_1^- = T_1^- I_0^- + R_1^- I_1^+ + S_1^-$$

$$I_1^+ = T_2^+ I_2^+ + R_2^+ I_1^- + S_2^+ \qquad I_2^- = T_2^- I_1^- + R_2^- I_2^+ + S_2^-$$

where layer 1 is above layer 2 and the incident radiance vectors on the combined layers are I_0^- downwelling on the top of layer 1 and I_2^+ upwelling on the bottom of layer 2. To get the adding formula the radiance vectors in the interface between layers 1 and 2 are eliminated.

Since we are not interested in the adding formula for the source vectors we can set them to zero, which does not affect the combined reflection and transmission. Substituting in for I_1^+ and I_1^- gives

$$I_1^- = T_1^- I_0^- + R_1^- (T_2^+ I_2^+ + R_2^+ I_1^-)$$

$$I_1^+ = T_2^+ I_2^+ + R_2^+ (T_1^- I_0^- + R_1^- I_1^+)$$

Solving for I_1^- and I_1^+ gives

$$I_1^- = \Gamma^- [T_1^- I_0^- + R_1^- T_2^+ I_2^+] \qquad \Gamma^- = [1 - R_1^- R_2^+]^{-1}$$

$$I_1^+ = \Gamma^+ [T_2^+ I_2^+ + R_2^+ T_1^- I_0^-] \qquad \Gamma^+ = [1 - R_2^+ R_1^-]^{-1}$$

Substituting in for I_1^- and I_1^+ in the upwelling layer 1 and the downwelling layer 2 interaction principle equations gives

$$I_0^+ = T_1^+ \Gamma^+ [T_2^+ I_2^+ + R_2^+ T_1^- I_0^-] + R_1^+ I_0^-$$

$$I_2^- = T_2^- \Gamma^- [T_1^- I_0^- + R_1^- T_2^+ I_2^+] + R_2^- I_2^+$$

which can be rearranged to

$$I_0^+ = [T_1^+ \Gamma^+ T_2^+] I_2^+ + [R_1^+ + T_1^+ \Gamma^+ R_2^+ T_1^-] I_0^-$$

$$I_2^- = [T_2^- \Gamma^- T_1^-] I_0^- + [R_2^- + T_2^- \Gamma^- R_1^- T_2^+] I_2^+$$

Therefore, the combined reflection and transmission matrices are

$$R_T^+ = R_1^+ + T_1^+ \Gamma^+ R_2^+ T_1^- \qquad T_T^+ = T_1^+ \Gamma^+ T_2^+$$

$$R_T^- = R_2^- + T_2^- \Gamma^- R_1^- T_2^+ \qquad T_T^- = T_2^- \Gamma^- T_1^-$$